PERTH MODERN SCHOOL
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Mathematics Specialist Year 11

Student name: MARKING KEY Teacher name: ______________

Date: Monday 10 August 2020

Task type: **Response + Investigation**

Time allowed: **45 minutes (for the entire booklet)**

Number of questions: **5**

Materials required: Calculator with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **38 marks**

Task weighting: **14% combined (8% for Test 2 and 6% for investigation 2)**

Formula sheet provided: **Yes**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Question 1 {1.3.4, 1.3.5} **(4 marks)**

(a) Let $x \in \mathbb{R}$. Prove that that $x^2 > x$ is false by giving a counterexample. (1 mark)

If $x = \frac{1}{2}$ ✓ gives a valid counterexample

then $x^2 = \frac{1}{4}$

which is $< \frac{1}{2}$

(b) Disprove the following statement: There exists $x \in \mathbb{R}$ such that $5 + x^2 = 1 - x^2$ (3 marks)

Negation: For all $x \in \mathbb{R}$, $5 + x^2 \neq 1 - x^2$

✓ negation

suppose that $5 + x^2 = 1 - x^2$

$2x^2 = -4$

$x^2 = -2$

✓ ✓ proves negation is true

which is impossible since $x^2 \geq 0$

The negation of the statement is true

$\therefore$ The given statement is false

accept any valid proof

Question 2 {2.1.1} **(5 marks)**

Solve $2\,cos\left(2(x+\frac{\pi}{3})\right) = -1$ given that $x \in [0, 2\pi]$. Show your working.

$\cos\left[2\left(x+\frac{\pi}{3}\right)\right] = -\frac{1}{2}$, $x \in [0, 2\pi]$

✓ rearranges

$x+\frac{\pi}{3} \in \left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$

$2\left(x+\frac{\pi}{3}\right) \in \left[\frac{2\pi}{3}, \frac{14\pi}{3}\right]$

$\therefore\ 2\left(x+\frac{\pi}{3}\right) = \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi+\frac{2\pi}{3}, 2\pi+\frac{4\pi}{3}, 4\pi+\frac{2\pi}{3}$

since $2\left(x+\frac{\pi}{3}\right) \in \left[\frac{2\pi}{3}, \frac{14\pi}{3}\right]$, ✓✓ first two solutions

$x+\frac{\pi}{3} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\therefore\ x = 0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$ as $x \in [0, 2\pi]$

✓ all solutions ✓ within specified domain

Question 3 {2.3.4, 2.3.6} **(8 marks)**

Use mathematical induction to prove that that $4^n + 6n - 1$ is divisible by 3 for all $n \in \{1,2,3,...\}$

✓ proposition

Let $P(n)$ be the proposition $4^n + 6n - 1$ is divisible by 3

For $P(1)$: $4^1 + 6(1) - 1 = 9$ ✓ prove $P(1)$

which is divisible by 3 states

$\therefore P(1)$ is true * ✓ $P(1)$ and $P(k+1)$ true

Assume that $P(k)$ is true ✓ assumption for $P(k)$

i.e. $4^k + 6k - 1 = 3m, \quad m \in \mathbb{Z}$

To prove $P(k+1)$ is true ✓ shows $P(k+1)$

i.e. $4^{k+1} + 6(k+1) - 1$ is divisible by 3

✓ uses $P(k)$

we have $4^{k+1} + 6k + 6 - 1 = 4^{k+1} + 6k - 1 + 6$

$= 4 \times 4^k + (3m - 4^k) + 6$

$= 4 \times 4^k - 4^k + 3m + 6$

$= 3 \times 4^k + 3m + 6$

$= 3(4^k + m + 2)$ ✓ proves $P(k+1)$

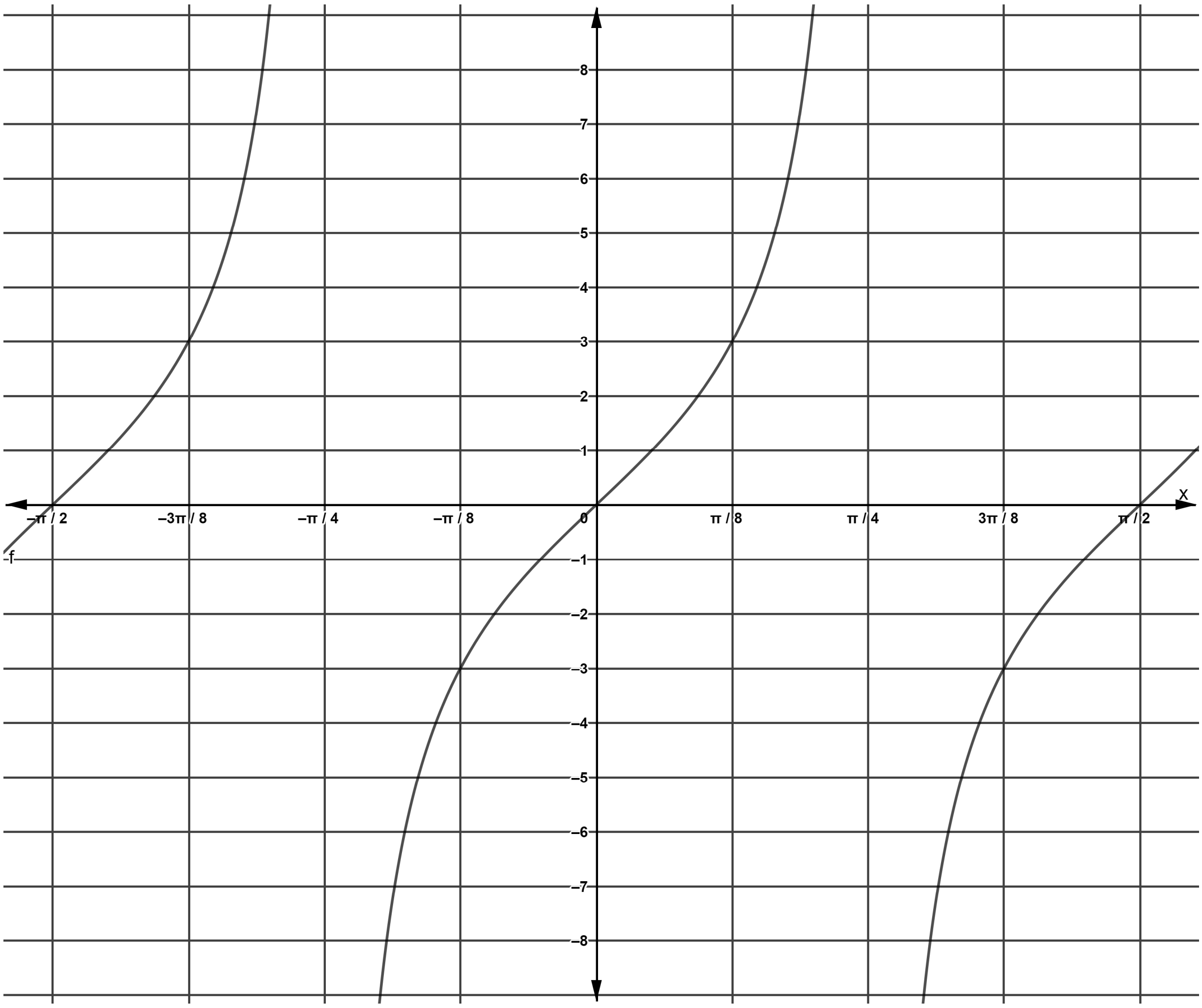
$\therefore 4^{k+1} + 6(k+1) - 1$ is divisible by 3

Thus $P(k+1)$ is true *

Hence by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$ ✓ concluding statement

Question 4 {2.1.2} **(6 marks)**

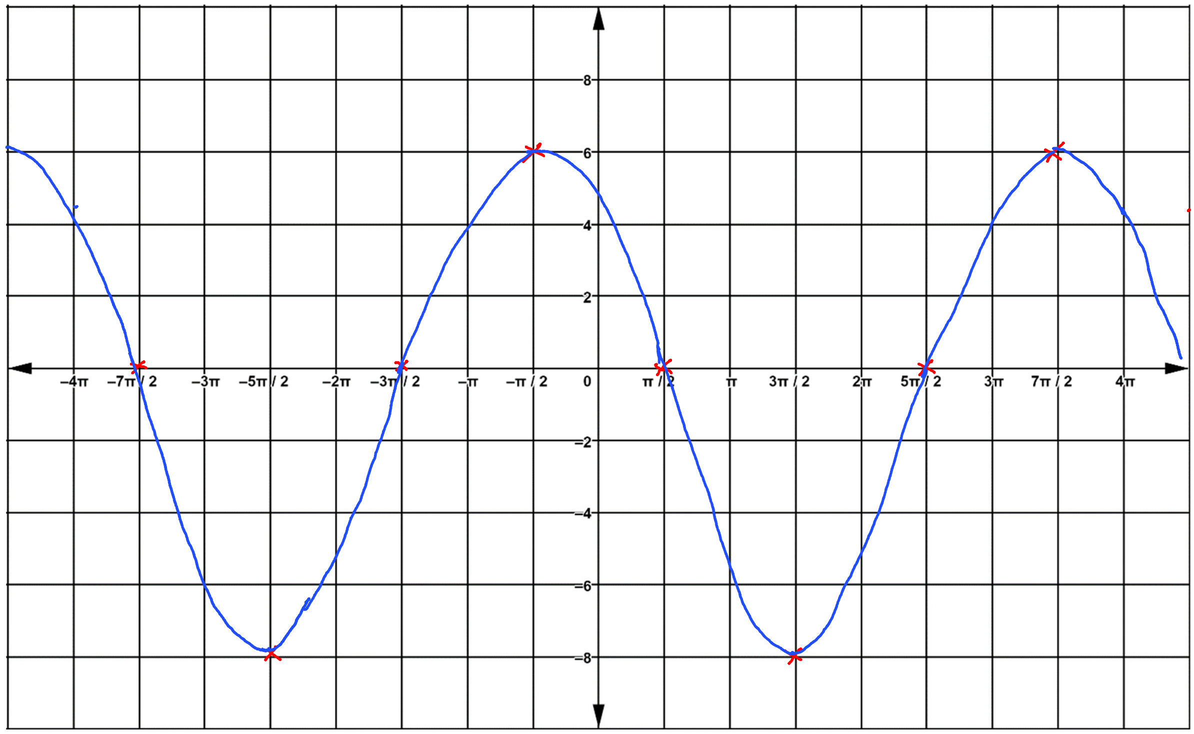
(a) The function $f(x) = a \tan(bx)$ has been graphed below. Determine the values of the constants a and b. (2 marks)



$a = 3$ ✓

$b = 2$ ✓ working not required

(b) Sketch the graph of $y = 6\,cos\left(\frac{1}{2}x + \frac{\pi}{4}\right)$. (4 marks)



✓ x-intercepts

✓ maxima & minima

✓ domain at least $\left[-\frac{7\pi}{2}, \frac{7\pi}{2}\right]$

✓ shape

Investigation Validation {2.1.3, 2.3.4, 2.3.5} **(15 marks)**

a) Use the identity

$$2sinAcosB = sin(A+B) + sin(A-B)$$

(or otherwise) to show that

$$2\,sin[x]\,cos[(2k+1)\,x] = sin[2(k+1)x] - sin\,[2kx]$$

(3 marks)

✓ sub

$2\sin(x)\cos[(2k+1)x] = \sin[x+(2k+1)x] + \sin[x-(2k+1)x]$

$= \sin[2kx + 2x] + \sin[-2kx]$

$= \sin[2kx+2x] - \sin[2kx]$

$= \sin[(2(k+1)x] - \sin[2kx]$

✓ simplify & factorise

✓ uses $\sin(-\alpha) = -\sin\alpha$

b) Given that $sin(x) \neq 0$ prove, by mathematical induction, that for all positive integers *n*,

$$cos(x) + cos(3x) + \cdots + cos[(2n-1)x] = \frac{sin\,(2nx)}{2sin\,(x)}$$

You may find the identity $sin(2A) = 2sinAcosA$ useful.

(12 marks)

let $P(n)$ be the proposition ✓ proposition

$$cos(x) + cos(3x) + \ldots + cos[(2n-1)x] = \frac{sin(2nx)}{2sin(x)}$$

Consider $P(1)$: LHS $= cos(x)$

$$RHS = \frac{sin(2x)}{2sinx}$$ ✓ prove $P(1)$

$$= \frac{2sinx\cos x}{2sinx}$$

$$= cosx$$

$= LHS$ $\therefore P(1)$ is true ✓ states $P(1)$ true

Assume $P(k)$ is true ✓ assumes

i.e. $cos(x) + cos(2x) + \ldots + cos[(2k-1)x] = \dfrac{sin(2kx)}{2sin(x)}$

To prove $P(k+1)$ is true ✓ shows $P(k+1)$ including RHS

i.e. $cos(x) + cos(2x) + \ldots + cos[(2k-1)x] + cos[(2(k+1)-1)x] = \dfrac{sin(2(k+1)x)}{2sinx}$

$LHS = \dfrac{sin(2kx)}{2sin(x)} + cos[(2k+1)x]$ ✓ subs LHS of $P(k)$

✓ common denominator

$$= \frac{sin(2kx) + 2sin(x)cos[(2k+1)x]}{2sin(x)}$$

uses part (a) or simplifies otherwise ✓ ✓

$$= \frac{sin(2kx) + sin[2(k+1)x] - sin(2kx)}{2sin(x)} = \frac{sin[2(k+1)x]}{2sin(x)}$$

$= RHS$ $\therefore P(k+1)$ is true ✓ states $P(k+1)$ true

Hence by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$ ✓ concluding statement

✓ uses LHS & RHS

(additional working space)